Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Sound and Vibration 279 (2005) 1155–1169
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Formulae for frequencies and modes of in-plane vibrations of small-sag inclined cables

Q. Wu, K. Takahashi*, S. Nakamura

Department of Civil Engineering, Faculty of Engineering, Nagasaki University, 1-14 Bunkyo-machi, Nagasaki 852-8521, Japan

Received 30 May 2003; accepted 12 January 2004

1. Introduction

The study of natural frequencies of cables and their modes of vibration has a long history. Among the more valuable recent contributions are those of Irvine and Caughey [1], who presented correct expressions and extensive numerical data for the natural frequencies and vibration modes of a uniform horizontal cable with small sag. The main advantage of this work is that, using one suitably dimensionless parameter, the natural frequencies of horizontal cables can be expressed in straight format. As a result, this approach provides a quick and accurate means of determining the natural frequencies of a horizontal cable. Moreover, Irvine [2,3] extended this solution to the case of an inclined cable. Taking the static profile of an inclined cable, he derived expressions for its dimensionless in-plane natural frequencies. Since the geometric parameter of a small-sag inclined cable is so small in actual structural applications, he concluded that this term could be ignored in calculating in-plane natural frequencies; that is, the properties of an inclined cable are taken to be the same as those of a horizontal cable.

In other research on inclined cables, Yamaguchi and Ito [4], Yamaguchi [5], and Triantafyllou [6] found that the in-plane natural vibration properties of an inclined cable differ from those of a horizontal cable in other respects. Yamaguchi and Ito [4] derived the basic equation of motion of an inclined cable in the global coordinate system using an accurate hyperbolic function to

*Corresponding author. Tel.: +81-95-819-2610 Fax: +81-95-819-2627.
E-mail address: takahasi@civil.nagasaki-u.ac.jp (K. Takahashi).

doi:10.1016/j.jsv.2004.01.004

simulate its static profile, and on this basis discussed additional in-plane vibration properties of an inclined cable. Triantafyllou [6] derived an asymptotic solution for a small-sag inclined cable and demonstrated the same additional in-plane vibration properties. These additional properties are that (i) there is no crossover of natural frequencies in the symmetric mode toward the natural frequencies of the antisymmetric mode and (ii) the corresponding modes are neither symmetric nor antisymmetric. Focusing on these phenomena, Triantafyllou and Grinfogel [7] also derived an asymptotic equation for the in-plane natural frequencies and modal shapes of a taut inclined cable.

This note presents a modification of the expressions for the in-plane natural frequencies of an inclined cable as derived by Irvine [2,3]. Beginning with Irvine's static profile for an inclined cable and considering the influence of inclination angle on in-plane natural frequencies and modal shapes, a simple and approximate modification of the Irvine's equations is derived that can demonstrate the additional in-plane vibration properties exhibited by an inclined cable. Here, the equations of motion of an inclined cable in the local coordinate system are induced first. The modified Irvine equations for in-plane natural frequencies and modal shapes of a small-sag inclined cable are then derived. Further, solutions given by the modified Irvine equations are compared with exact results obtained by a Galerkin method, and the applicable range of the new equations is discussed in detail.

2. Equations of motion of an inclined cable

An inclined cable with a uniform cross-section and uniform weight per unit length hanging between two points is considered, as shown in Fig. 1. In this note, the horizontal and vertical axes (x,z) are treated as the global coordinate system, while the longitudinal and transverse coordinates (x^*, z^*) are treated as the local coordinate system.

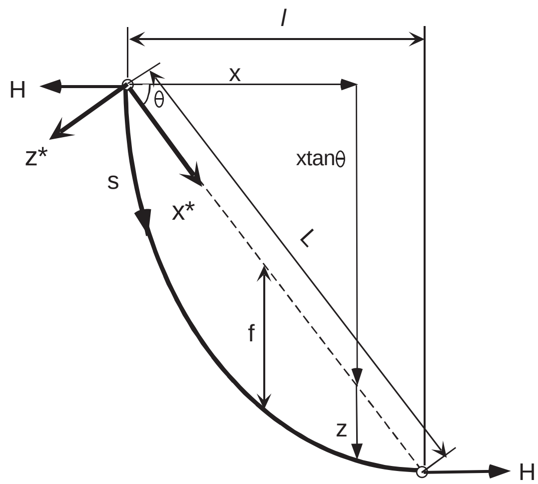

Fig. 1. Geometry of an inclined cable.

2.1. Static profile of an inclined cable

In the local coordinate system (x^*, z^*), the equilibrium of an inclined cable can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x^*}{\mathrm{d}s}\right) = -mg\sin\theta, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}z^*}{\mathrm{d}s}\right) = -mg\cos\theta, \tag{2}$$

where T is the initial tension, s is the coordinate along the cable, θ is the inclination angle, m is the mass per unit length of the cable and g is the gravitational acceleration.

From Eqs. (1) and (2), the following equations can be obtained:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x^*}{\mathrm{d}s}\cos\theta - T\frac{\mathrm{d}z^*}{\mathrm{d}s}\sin\,\theta\right) = 0, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x^*}{\mathrm{d}s}\sin\theta + T\frac{\mathrm{d}z^*}{\mathrm{d}s}\cos\theta\right) = -mg. \tag{4}$$

Eq. (3) may be integrated directly to

$$H = T\frac{\mathrm{d}x^*}{\mathrm{d}s}\cos\theta - T\frac{\mathrm{d}z^*}{\mathrm{d}s}\sin\theta = \text{Const.}, \tag{5}$$

where H is the horizontal component of cable tension which is constant everywhere. Consequently Eq. (4) may be derived

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}z + \mathrm{d}x\tan\theta}{\mathrm{d}s}\right) = -mg, \tag{6}$$

where $x = x^*\cos\theta - z^*\sin\theta$ and $z = z^*/\cos\theta$.

Retaining only the first-order terms in $\mathrm{d}z/\mathrm{d}x$ in Eq. (6) and solving the resulting differential equation by the method of successive approximation, the static profile of an inclined cable is obtained as

$$\bar{\bar{z}} = \tfrac{1}{2}\bar{x}(1-\bar{x})\left\{1 + \frac{\varepsilon}{6}(1-2\bar{x})\right\} + O(\varepsilon^2), \tag{7}$$

where $\bar{\bar{z}} = \bar{z}/(8\beta\cos\theta)$, $\bar{z} = z/L$, $\bar{x} = x/L$, $\beta = mgL/(8H\sec\theta)$, $\varepsilon = mgL/(H\sec\theta)\sin\theta = 8\beta\sin\theta$ and L is the span of the cable between two supports.

If $x = x^*\cos\theta - z^*\sin\theta$ and $z = z^*/\cos\theta$ are adopted, the static profile of an inclined cable in the local coordinate system (x^*, y^*) is also found by ignoring square and cubic terms of ε as

$$\bar{\bar{z}}^* = \tfrac{1}{2}\bar{x}^*(1-\bar{x}^*)\left\{1 - \frac{\varepsilon}{3}(1-2\bar{x}^*)\right\} + O(\varepsilon^2), \tag{8}$$

where $\bar{\bar{z}}^* = \bar{z}/(8\beta\cos\theta)$, $\bar{z}^* = z^*/L$ and $\bar{x}^* = x^*/L$.

Eqs. (7) and (8) are the same as the static profile derived by Irvine [2,3]. Irvine changed the notation from ε to ε^* in Eq. (8), while notation ε is unchanged here.

Moreover, according to Ref. [2], the ratio of cable sag ($d^* = mgL^2\cos\theta/(8H)$ in $\bar{x}^* = \frac{1}{2}$ by neglecting ε) to cable length L is $\beta\cos\theta$. Here $\beta\cos\theta$ is called the ratio of sag to span in the local coordinate system (x^*, z^*).

2.2. Equations of in-plane motion for an inclined cable

In the local coordinate system (x^*, z^*), the displacement of a cable element under in-plane vibrations is shown in Fig. 2. Equations of in-plane motion of an inclined cable are obtained from Fig. 2 as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((T+\tau)\left(\frac{\mathrm{d}x^*}{\mathrm{d}s}+\frac{\partial u^*}{\partial s}\right)\right) = m\frac{\partial^2 u^*}{\partial t^2} - mg\sin\theta - p_{x^*}(x^*, t), \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((T+\tau)\left(\frac{\mathrm{d}z^*}{\mathrm{d}s}+\frac{\partial w^*}{\partial s}\right)\right) = m\frac{\partial^2 w^*}{\partial t^2} - mg\cos\theta - p_{z^*}(x^*, t), \tag{10}$$

where τ is the additional tension generated, w^* is the transverse displacement in the z^* direction, u^* is the longitudinal displacement in the x^* direction, $p_{x^*}(x^*, t)$ and $p_{z^*}(x^*, t)$ are loads in the x^* and z^* directions and t is time.

Removing the self-weight term using Eqs. (1) and (2), the equations of in-plane motion for an inclined cable in the local coordinate system (x^*, z^*) are obtained as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\tau\frac{\mathrm{d}x^*}{\mathrm{d}s}+(T+\tau)\frac{\partial u^*}{\partial s}\right) = m\frac{\partial^2 u^*}{\partial t^2} - p_{x^*}(x^*, t), \tag{11}$$

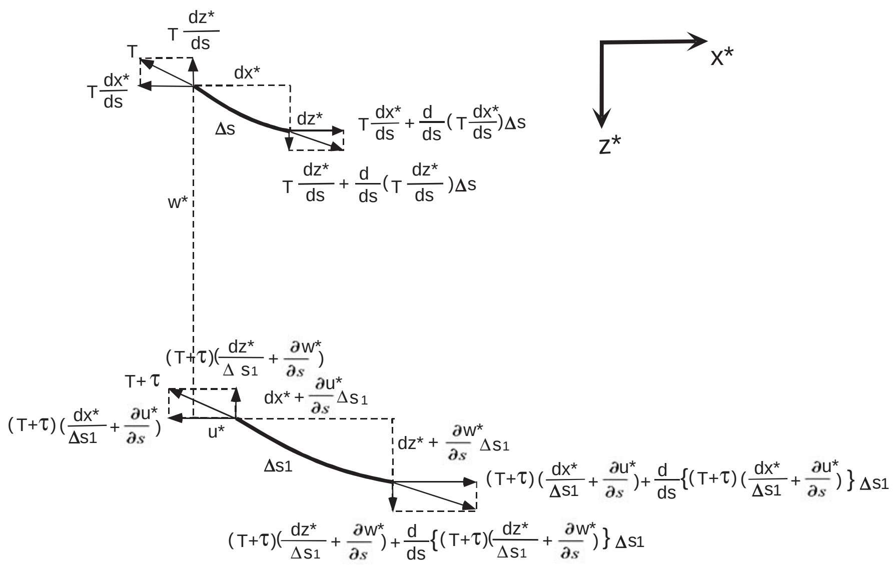

Fig. 2. Displacements of an element of the cable in the local coordinate system.

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\tau\frac{\mathrm{d}z^*}{\mathrm{d}s}+(T+\tau)\frac{\partial w^*}{\partial s}\right)=m\frac{\partial^2 w^*}{\partial t^2}-p_{z^*}(x^*,t). \tag{12}$$

Substituting $u=u^*\cos\theta-w^*\sin\theta$, $w=u^*\sin\theta+w^*\cos\theta$, $p_x=p_{x*}\cos\theta-p_{y^*}\sin\theta$ and $p_y=p_{x^*}\sin\theta+p_{y^*}\cos\theta$ into Eqs. (11) and (12) yields

$$\frac{\partial}{\partial s}\left\{\tau\frac{\mathrm{d}x}{\mathrm{d}s}+(T+\tau)\frac{\partial u}{\partial s}\right\}=m\frac{\partial^2 u}{\partial t^2}-p_x(x,t), \tag{13}$$

$$\frac{\partial}{\partial s}\left\{\tau\frac{\mathrm{d}x\tan\theta+\mathrm{d}z}{\mathrm{d}s}+(T+\tau)\frac{\partial w}{\partial s}\right\}=m\frac{\partial^2 w}{\partial t^2}-p_y(x,t), \tag{14}$$

where u and w are displacements in the x and z directions, and p_x and p_z are loads in the x and z directions, respectively.

Eqs. (13) and (14) represent the equations of motion of an inclined cable in the global coordinate system (x, z), and they are the same as those derived by Yamaguchi and Ito [4]. This confirms the correctness of the equations of motion in the local coordinate system (x^*, z^*).

2.3. Modified expressions for in-plane modal shapes and natural frequencies of a flat-sag inclined cable

Using $h^*=\tau\mathrm{d}x^*/\mathrm{d}s$ and $H^*=T\,\mathrm{d}x^*/\mathrm{d}s=H\sec\theta/(1-(\mathrm{d}z^*/\mathrm{d}x^*)\tan\theta)\approx H\sec\theta$, Eqs. (11) and (12) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(h^*+(H^*+h^*)\frac{\partial u^*}{\partial x^*}\right)=m\frac{\partial^2 u^*}{\partial t^2}-p_{x^*}(x^*,t), \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(h^*\frac{\mathrm{d}z^*}{\mathrm{d}x^*}+(H^*+h^*)\frac{\partial w^*}{\partial x^*}\right)=m\frac{\partial^2 w^*}{\partial t^2}-p_{z^*}(x^*,t). \tag{16}$$

If only a flat-sag cable is considered, the longitudinal component of the equation of motion may be considered unimportant and can be dropped. Furthermore, $\mathrm{d}s\approx\mathrm{d}x^*$ is assumed and h^* is the same everywhere. Eq. (16) can then be reduced to

$$h^*\frac{\mathrm{d}^2 z^*}{\mathrm{d}x^{*2}}+(H^*+h^*)\frac{\partial^2 w^*}{\partial x^{*2}}=m\frac{\partial^2 w^*}{\partial t^2}-p_{z^*}(x^*,t), \tag{17}$$

where h^* is the additional horizontal component of tension and is a function of time alone [3]. From Fig. 2, the additional tension generated τ is given by

$$\tau=EA\left\{\frac{\mathrm{d}z^*}{\mathrm{d}s}\frac{\partial w^*}{\partial s}+\frac{\mathrm{d}x^*}{\mathrm{d}s}\frac{\partial u^*}{\partial s}+\frac{1}{2}\left(\frac{\partial w^*}{\partial s}\right)^2+\frac{1}{2}\left(\frac{\partial u^*}{\partial s}\right)^2\right\}, \tag{18}$$

where E is Young's modulus and A is the cross-sectional area.

Considering the first-order terms of Eq. (18) and adopting $h^* = \tau\, \mathrm{d}x^*/\mathrm{d}s$, the additional transverse tension h^* is obtained as

$$h^* = EA\left(\frac{\mathrm{d}x^*}{\mathrm{d}s}\right)^3\left(\frac{\partial u^*}{\partial x^*} + \frac{dz^*}{dx^*}\frac{\partial w^*}{\partial x^*}\right) \tag{19}$$

which we integrate into the form

$$h^* = \frac{EA}{Le}\int_0^L \frac{\mathrm{d}^2 z^*}{\mathrm{d}x^{*2}} w^*\,\mathrm{d}x^*, \tag{20}$$

where $Le = \int_0^L (\mathrm{d}s/\mathrm{d}x^*)^3 \mathrm{d}x^*$ is the length of the cable.

Eq. (17) may be simplified as follows by considering the linear terms and neglecting load p_{z^*}:

$$h^*\frac{\mathrm{d}^2 z^*}{\mathrm{d}x^{*2}} + H^*\frac{\partial^2 w^*}{\partial x^{*2}} = m\frac{\partial^2 w^*}{\partial t^2}. \tag{21}$$

Eqs. (20) and (21) constitute a linear homogeneous system in w^*. With these two equations the fundamental features of linear theory of free vibrations may be explored.

By making Eqs. (20) and (21) non-dimensional, the following equations of in-plane motion are obtained as

$$\bar{h}^*\frac{\mathrm{d}^2\bar{\bar{z}}^*}{\mathrm{d}\bar{x}^{*2}} + \frac{\partial^2\bar{\bar{w}}^*}{\partial\bar{x}^{*2}} = \pi^2\frac{\partial^2\bar{\bar{w}}^*}{\partial\tau^2}, \tag{22}$$

$$\bar{h}^* = \frac{1}{\lambda^2}\int_0^L \frac{\mathrm{d}^2\bar{\bar{z}}^*}{\mathrm{d}\bar{x}^{*2}}\bar{\bar{w}}^*\,dx^* \tag{23}$$

where $\bar{\bar{w}}^* = \bar{w}^*/(8\beta\cos\theta)$ $(\bar{w}^* = w^*/L)$ is the non-dimensional displacement in the z^* direction, $\bar{x}^* = x^*/L$ is the non-dimensional coordinate in the x^* direction, $\Delta\bar{H} = \Delta H/(H\sec\theta)$, $\tau = \omega_0 t$ is the non-dimensional time, $\omega_0 = \sqrt{H\sec\theta/m(\pi/L)^2}$ is the first natural circular frequency of the inclined taut string, $\lambda^2 = k^2(8\beta\cos\theta)^2/Le$ is Irvine parameter [2,3], $k^2 = EA/(H\sec\theta)$ is the ratio of the elongation stiffness to horizontal component of tension [4] and $\overline{Le} = 1 + (8\beta\cos\theta)^2/8$ [2,3].

Using Eq. (8) and $\bar{\bar{w}} = \tilde{w}(\bar{x}^*)e^{i\omega\tau}$, Eq. (22) becomes

$$\frac{\partial^2\tilde{w}^*}{\partial\bar{x}^{*2}} + \pi^2\omega^2\tilde{w}^* = \bar{h}^*(1-\varepsilon+2\varepsilon\bar{x}^*). \tag{24}$$

A solution to Eq. (24) that satisfies zero boundary conditions at two supports is

$$\bar{\bar{w}}^* = \frac{\bar{h}^*}{\pi^2\omega^2}\left\{(1-\varepsilon+2\varepsilon\bar{x}^*) - \left(\tan\frac{\pi\omega}{2} + \frac{\varepsilon}{\tan(\pi\omega/2)}\right)\sin\pi\omega\bar{x}^* - (1-\varepsilon)\cos\pi\omega\bar{x}^*\right\}. \tag{25}$$

Eq. (25) is now used to eliminate $\bar{h}^*$ and obtain the following transcendental equation:

$$\frac{4}{\lambda^2}\left(\frac{\pi\omega}{2}\right)^3 = \left(1+\frac{\varepsilon^2}{3}\right)\frac{\pi\omega}{2} - \tan\frac{\pi\omega}{2} + \frac{\varepsilon^2}{\tan\pi\omega/2} - \frac{2\varepsilon^2}{\pi\omega}. \tag{26}$$

Eqs. (25) and (26) are the modified Irvine equations for the in-plane modal shapes and natural frequencies of an inclined cable.

2.4. Comparisons with Irvine equation

Irvine thought that parameter ε could be ignored despite having derived the static profile of an inclined cable in consideration of ε [2,3]. Taking this approach, Eqs. (25) and (26) become

$$\bar{\bar{w}}^* = \frac{\bar{h}^*}{\pi^2\omega^2}\left\{1 - \tan\frac{\pi\omega}{2}\sin\pi\omega\bar{x}^* - \cos\pi\omega\bar{x}^*\right\}, \tag{27}$$

$$\frac{4}{\lambda^2}\left(\frac{\pi\omega}{2}\right)^3 = \frac{\pi\omega}{2} - \tan\frac{\pi\omega}{2}. \tag{28}$$

Comparing these Irvine equations with the modified Irvine equations derived here, the differences are that modal shapes are influenced by parameter ε and natural frequencies are influenced by ε^2.

The greatness of Irvine's work is to offer a simple and accurate equation for the in-plane natural frequencies and modal shapes of a horizontal cable based on very few assumptions. So these modified Irvine equations are based on the same assumptions as for a horizontal cable. Moreover, the introduction of an inclination angle results in equations including the parameter ε besides the Irvine parameter λ^2.

Irvine used the single parameter λ^2 to calculate in-plane natural frequencies and modal shapes for a horizontal cable [1,3]. Triantafyllou and Grinfogel adopted two parameters λ^2 and ε in considering the in-plane vibration properties of an inclined cable [6,7]. From the definitions $\lambda^2 = k^2(8\beta\cos\theta)^2/(1 + (8\beta\cos\theta)^2/8)$ and $\varepsilon = 8\beta\sin\theta$, parameters λ^2 and ε are seen to be related to three parameters: β, k^2 and θ. That is, the natural frequencies of an inclined cable actually depend on these three parameters, as pointed out by Henghold et al. [8]. In this note, in order to compare the results obtained by a Galerkin method, the following discussion is based on the three parameters $\beta\cos\theta$, k^2 and θ.

3. Numerical results

3.1. In-plane free vibration analysis by a Galerkin method

In order to check the accuracy of the modified Irvine equations, a Galerkin method [4,5] is applied to Eqs. (11) and (12) by assuming

$$\bar{u}^* = \sum_{i=1}^{\infty} P^i_{x^*}(\tau)\sin\frac{i\pi\bar{s}}{\bar{s}_t}, \bar{w}^* = \sum_{i=1}^{\infty} P^i_{y^*}(\tau)\sin\frac{i\pi\bar{s}}{\bar{s}_t}, \tag{29}$$

where $P^i_{x^*}(\tau)$ and $P^i_{z^*}(\tau)$ are unknown functions of time, $\bar{s}_t = s_t/L$ is the non-dimensional length of the cable and $\bar{s} = s/L$.

Non-dimensional Eqs. (11) and (12) (here we neglect the loads) become

$$\ddot{P}^i_{x^*}(\tau) + \frac{1}{\pi^2}\sum_{i=1}^{\infty}\frac{2ij\pi^2}{\bar{s}_t^3}\left\{\left(k^2I_2^* + I_1^*\right)P^i_{x^*}(\tau) + k^2I_3^*P^i_{y^*}(\tau)\right\} = 0, \tag{30}$$

$$\ddot{P}^i_{y^*}(\tau) + \frac{1}{\pi^2}\sum_{i=1}^{\infty}\frac{2ij\pi^2}{\bar{s}_t^3}\left\{\left(k^2I_4^* + I_1^*\right)P^i_{y^*}(\tau) + k^2I_3^*\cdot P^i_{x^*}(\tau)\right\} = 0, \tag{31}$$

where

$$I_1^* = \int_0^{\bar{s}_t} \frac{\cos\theta}{(\mathrm{d}x^*/\mathrm{d}\bar{s}^*)\cos\theta - (\mathrm{d}z^*/\mathrm{d}s^*)\sin\theta} \cos\frac{i\pi\bar{s}}{\bar{s}_t}\cos\frac{j\pi\bar{s}}{\bar{s}_t}\,\mathrm{d}s,$$

$$I_2^* = \int_0^{\bar{s}_t} \left(\frac{\mathrm{d}\bar{x}^*}{\mathrm{d}\bar{s}}\right)^2 \cos\frac{i\pi\bar{s}}{\bar{s}_t}\cos\frac{j\pi\bar{s}}{\bar{s}_t}\,\mathrm{d}s,$$

$$I_3^* = \int_0^{\bar{s}_t} \frac{\mathrm{d}\bar{x}^*}{\mathrm{d}\bar{s}} \cdot \frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{s}} \cos\frac{i\pi\bar{s}}{\bar{s}_t}\cos\frac{j\pi\bar{s}}{\bar{s}_t}\,\mathrm{d}\bar{s},$$

$$I_4^* = \int_0^{\bar{s}_t} \left(\frac{\mathrm{d}\bar{z}^*}{\mathrm{d}\bar{s}}\right)^2 \cos\frac{i\pi\bar{s}}{\bar{s}_t}\cos\frac{j\pi\bar{s}}{\bar{s}_t}\,\mathrm{d}\bar{s}.$$

The static profile of an inclined cable is assumed to be a hyperbolic curve described by

$$\bar{z} = \frac{-1}{8\beta}\cosh(-8\beta\bar{x} + c_1) - \bar{x}\tan\theta + \frac{1}{8\beta}\cosh c_1, \tag{32}$$

where

$$c_1 = \operatorname{arcsinh}\left(\frac{-4\beta\tan\theta}{\sinh(-4\beta)}\right) + 4\beta$$

The eigenvalue problem can be derived from the linearized governing equations of motion given by Eqs. (30) and (31), which can be discretized into a finite degree-of-freedom system by the generalized coordinate method. The in-plane natural frequencies and modal shapes of an inclined cable can be obtained numerically without any limitations as to the cable's static profile by solving the eigenvalue problem.

3.2. In-plane natural frequencies and modal shapes of an inclined cable

The parameter $\beta\cos\theta$ is set in the range from 0.001 to 0.125 and k^2 is 900. Figs. 3–6 show the non-dimensional in-plane natural frequencies of both horizontal cables and inclined cables with the inclination angle θ.

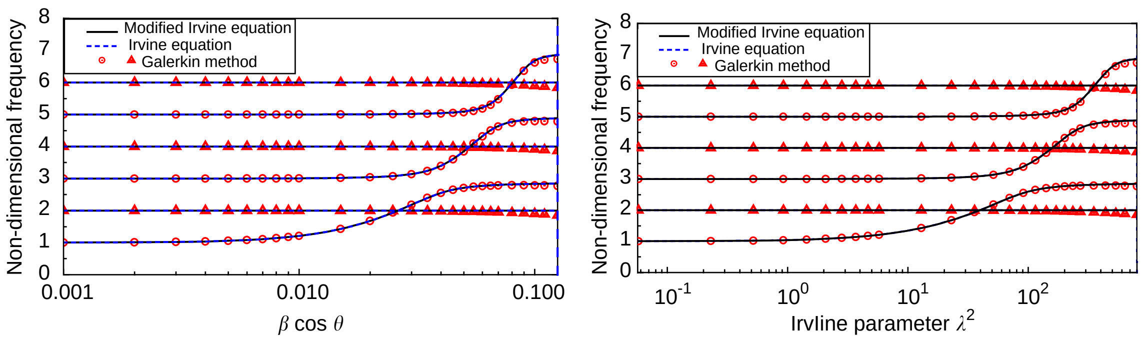

Fig. 3. In-plane natural frequencies of horizontal cables ($k^2 = 900$, $\theta = 0°$).

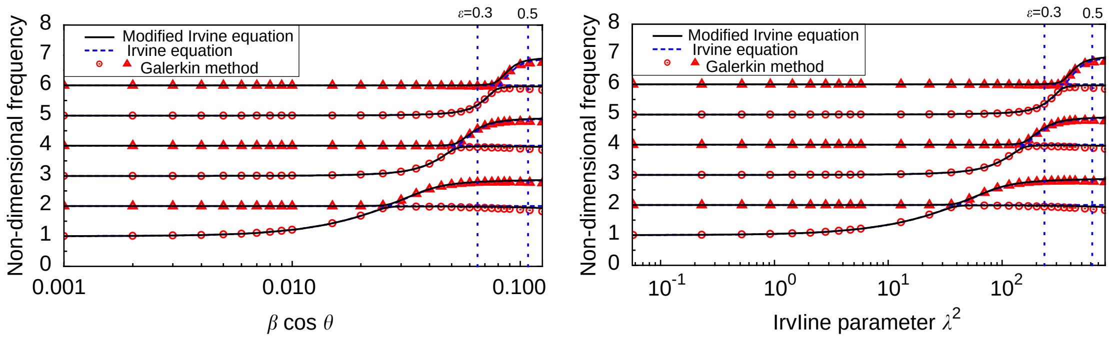

Fig. 4. In-plane natural frequencies of inclined cables ($k^2 = 900$, $\theta = 30°$).

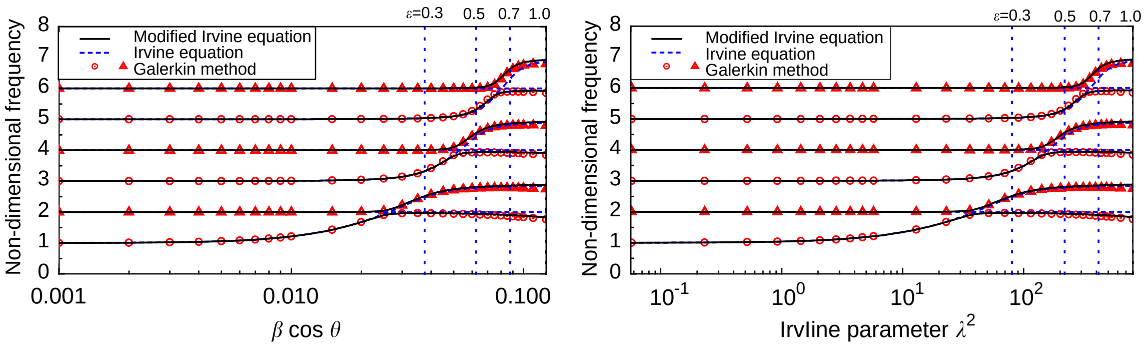

Fig. 5. In-plane natural frequencies of inclined cables ($k^2 = 900$, $\theta = 45°$).

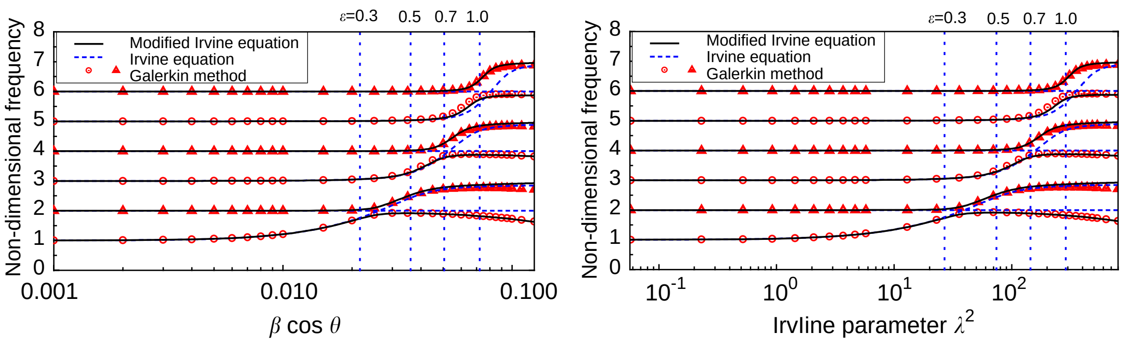

Fig. 6. In-plane natural frequencies of inclined cables ($k^2 = 900$, $\theta = 60°$).

For horizontal cables (inclination angle $\theta = 0°$ see Fig. 3), in-plane natural frequencies calculated using the modified Irvine equation and the original Irvine equation coincide well with the exact results obtained using the Galerkin method. This figure confirms that crossover of the

Table 1
In-plane natural frequencies of cables ($\beta \cos \theta = 0.015$, $\lambda^2 = 12.94$, $k^2 = 900$)

θ		1st	2nd	3rd	4th	5th	ε
	Irvine equation	1.43	2.00	3.02	4.00	5.00	
0°	Modified Irvine equation	1.43	2.00	3.02	4.00	5.00	0
	Galerkin method	1.43	2.00	3.02	4.00	5.00	
30°	Modified Irvine equation	1.43	2.00	3.02	4.00	5.00	0.07
	Galerkin method	1.43	2.00	3.02	4.00	5.00	
45°	Modified Irvine equation	1.43	2.00	3.02	4.00	5.00	0.12
	Galerkin method	1.43	2.00	3.02	4.00	5.00	
60°	Modified Irvine equation	1.43	2.00	3.02	4.00	5.00	0.21
	Galerkin method	1.43	2.00	3.02	4.00	5.00	

Table 2
In-plane natural frequencies of cables ($\beta \cos \theta = 0.026$, $\lambda^2 = 39.48$, $k^2 = 900$)

θ		1st	2nd	3rd	4th	5^{th}	ε
	Irvine equation	1.99	2.00	3.09	4.00	5.01	
0°	Modified Irvine equation	1.99	2.00	3.09	4.00	5.01	0
	Galerkin method	1.99	1.99	3.09	3.99	5.01	
30°	Modified Irvine equation	1.95	2.04	3.09	4.00	5.02	0.12
	Galerkin method	1.95	2.03	3.09	3.99	5.01	
45°	Modified Irvine equation	1.92	2.07	3.09	4.00	5.02	0.21
	Galerkin method	1.93	2.06	3.09	4.00	5.01	
60°	Modified Irvine equation	*1.87*	*2.13*	*3.09*	*4.00*	*5.02*	*0.36*
	Galerkin method	*1.88*	*2.11*	*3.10*	*4.00*	*5.02*	

Table 3
In-plane natural frequencies of cables ($\beta \cos \theta = 0.040$, $\lambda^2 = 91.00$, $k^2 = 900$)

θ		1st	2nd	3rd	4th	5th	ε
	Irvine equation	2.00	2.55	3.40	4.00	5.05	
0°	Modified Irvine equation	2.00	2.55	3.40	4.00	5.05	0
	Galerkin method	1.98	2.55	3.39	3.99	5.03	
30°	Modified Irvine equation	1.99	2.56	3.41	4.00	5.05	0.19
	Galerkin method	1.98	2.56	3.40	3.99	5.03	
45°	Modified Irvine equation	*1.97*	*2.58*	*3.41*	*4.01*	*5.05*	*0.32*
	Galerkin method	*1.96*	*2.57*	*3.42*	*4.00*	*5.04*	
60°	Modified Irvine equation	*1.91*	*2.64*	*3.43*	*4.04*	*5.05*	*0.55*
	Galerkin method	*1.92*	*2.60*	*3.48*	*4.03*	*5.06*	

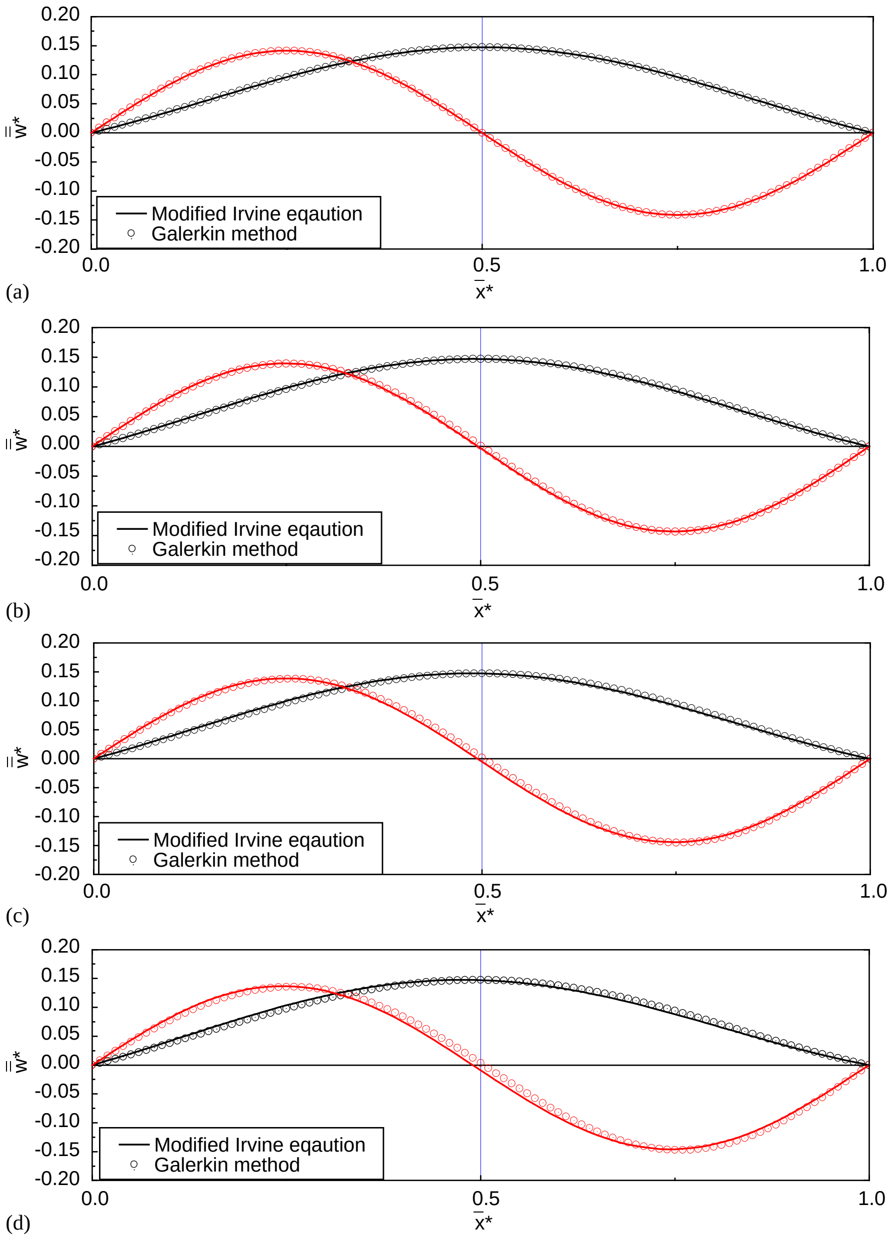

Fig. 7. In-plane natural modal shapes of cables ($\beta \cos\theta = 0.015$, $k^2 = 900$).

in-plane natural frequency of the symmetric mode occurs toward the natural frequency of the antisymmetric mode in the case of horizontal cables.

For inclined cables (inclination angle $\theta = 30°$, 45°, and 60°; see Figs. 4–6), in-plane natural frequencies given by the modified equations coincide well with those by the Galerkin method, and

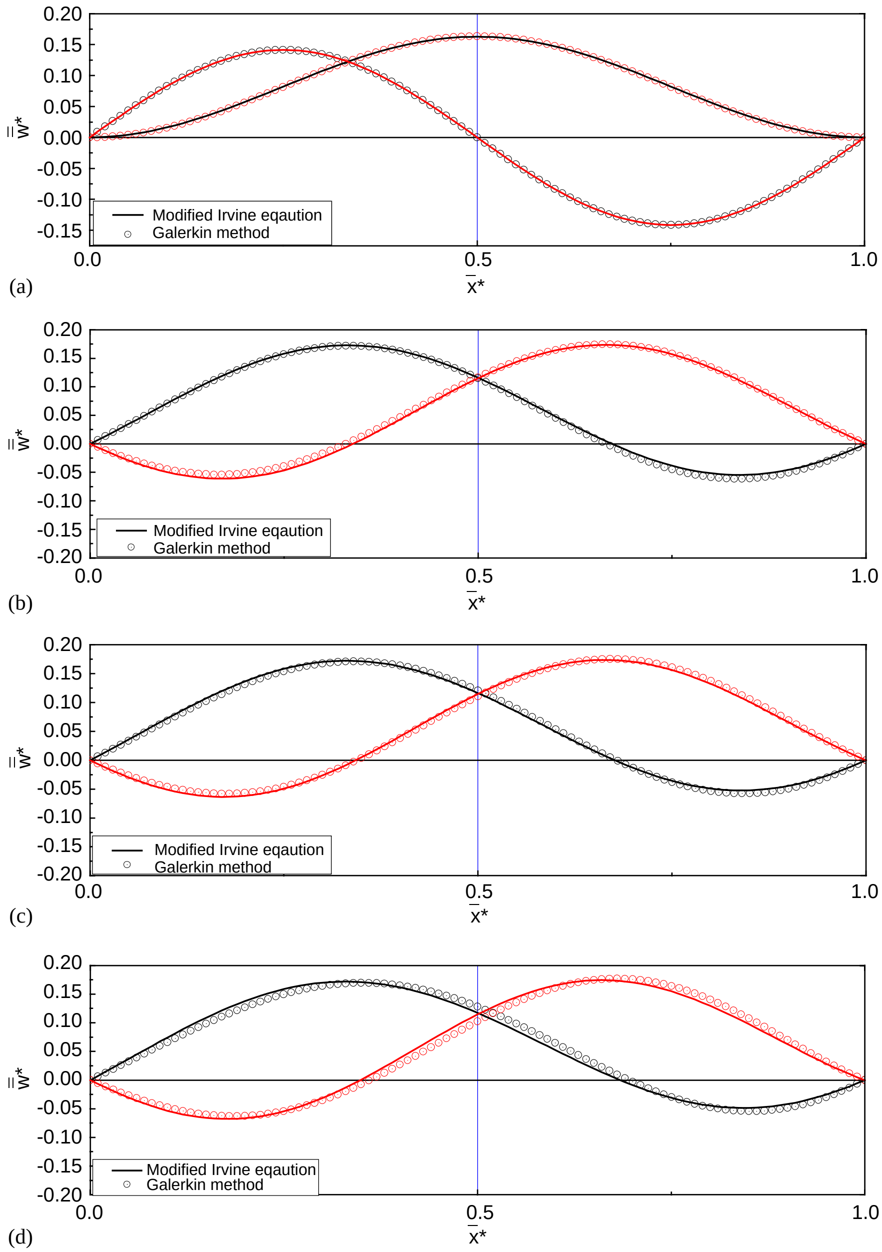

Fig. 8. In-plane natural modal shapes of cables ($\beta \cos\theta = 0.026$, $k^2 = 900$).

the additional properties of inclined cables are properly described. In contrast, the in-plane vibration properties of the inclined cables as given by the Irvine equation are the same as those of a horizontal cable. This discrepancy is the reason for the investigations of inclined cables by Yamaguchi and Ito [4] and Triantafyllou [6]. By taking ε^2 into account in the

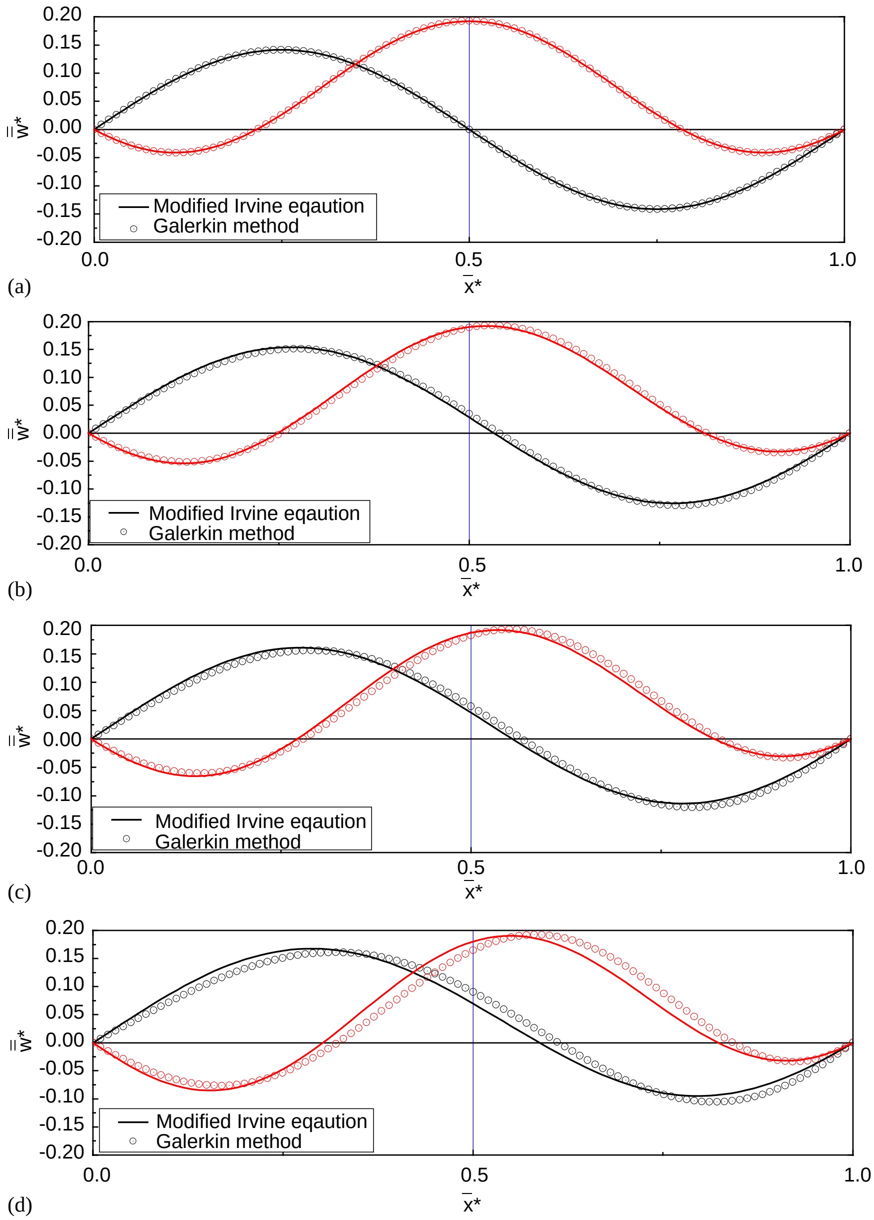

Fig. 9. In-plane natural modal shapes of cables ($\beta \cos \theta = 0.040$, $k^2 = 900$).

equation for the in-plane natural frequencies of an inclined cable (see Eq. (26)), it is possible to capture the additional properties of an inclined cable such that crossover of in-plane natural frequency of the symmetric mode never occurs toward the natural frequency of the antisymmetric mode.

Tables 1–3 show the in-plane natural frequencies of various cables, while Figs. 7–9 give the first two modal shapes of the same cables. The transverse components w^* given by the Galerkin method are also shown in these figures. Checking the results against the exact values obtained by the Galerkin method demonstrates that the in-plane natural frequencies and modal shapes coincide well.

The obtained modal shapes confirm that, by taking the parameter ε into account in the equation for the modal shape of an inclined cable (see Eq. (25)), an additional property of inclined cables can be properly captured; that is, the fact that the modes are neither symmetric nor antisymmetric when the cable has an inclination angle [6].

These results demonstrate that modified Irvine equations for an inclined cable, in which ε^2 is taken into account in calculating in-plane natural frequency and ε in calculating modal shape, offer a simple yet accurate means of expressing the properties of a cable with an inclination.

The limitations of these modified Irvine equations are $\beta \cos\theta < \frac{1}{8}$ and $\varepsilon < 1$. The broken lines in Figs 4–6 indicate ε =0.3, 0.5, 0.7 and 1.0 (the upper limit). The accuracy of the modified equations is satisfactory even if the parameter ε is not very small.

4. Concluding remarks

Two main conclusions may be drawn from this investigation, as follows:

(1) Equations for the in-plane motion of an inclined cable in the local coordinate system have been derived and their correctness verified through comparison with equations by Yamaguchi and Ito using the coordinate change method.
(2) These modified Irvine equations, which take into account ε^2 ($\varepsilon = 8\beta\cos\theta$) in in-plane natural frequency and ε in modal shape, were derived for the in-plane natural frequencies and modal shapes of an inclined cable with small sag based on Irvine's assumptions. The formulae not only correctly express the additional properties of an inclined cable, but can also be used to calculate in-plane natural frequencies and modal shapes correctly. Consequently, these simple, approximate formulae may prove useful in the design of cable structures.

References

[1] H.M. Irvine, T.K. Caughey, The linear theory of free vibrations of a suspended cable, *Proceedings of the Royal Society Series* A 341 (1974) 299–315.
[2] H.M. Irvine, Free vibrations of inclined cables, *Journal of the Structural Division* 104 (ST2) (1978) 343–347.
[3] H.M. Irvine, *Cable Structures*, The Massachusetts Institute of Technology Press, Cambridge, MA, 1981.
[4] H. Yamaguchi, M. Ito, Linear theory of free vibrations of an inclined cable in three dimensions, *Proceedings of The Japan Society of Civil Engineers* 286 (1979) 29–36 (in Japanese).
[5] H. Yamaguchi, Fundamentals of cable dynamics, *Proceedings of International Seminar on Cable Dynamics*, Technical Committee on Cable Structures and Wind, Japan Association for Wind Engineer, Tokyo, 1997, pp. 81–94.

[6] M.S. Triantafyllou, The dynamics of taut inclined cables, *Quarterly Journal of Mechanics and Applied Mathematics* 37(3) (1984) 421–440.
[7] M.S. Triantafyllou, L. Grinfogel, Natural frequencies and modes of inclined cables, *Journal of Structural Engineering* 112(1) (1986) 139–148.
[8] W.M. Henghold, J.J. Russell, J.D. Morgan, Free vibrations of cable in three dimensions, *Journal of the Structural Division* 103 (ST5) (1977) 1127–1136.